

Numerical Investigation of Blow-Up and Energy Decay Phenomena in Variable-Exponent PDE Models Using COMSOL Multiphysics®

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Abstract

In this study, we first consider a coupled Klein–Gordon system with a variable exponent, followed by a $p(x)$ -biharmonic equation subject to appropriate initial and boundary conditions. The main objective is to numerically examine the evolution of the solutions and illustrate the corresponding blow-up and energy decay phenomena. The numerical findings are consistent with the theoretical predictions, providing support for the qualitative analysis.

Keywords: Variable-exponent PDEs, Decay and Blow-up Phenomena, Numerical Investigation, COMSOL Multiphysics®.

Introduction

Partial differential equations with variable-exponent nonlinearities offer a flexible framework for modeling complex nonlinear phenomena in heterogeneous media, allowing the nonlinear response to vary with spatial position. This spatial variability leads to rich qualitative dynamics and raises important questions concerning the long-time behavior of solutions. In particular, understanding the stability, energy decay, and possible finite-time blow-up of solutions is essential for characterizing the underlying dynamics. In this work, we investigate two nonlinear partial differential equation models with variable exponents and analyze their dynamical behavior using COMSOL Multiphysics®. The first, which is a nonlinear coupled Klein–Gordon system, was implemented using the Wave Equation interface, whereas the second, $p(x)$ -biharmonic model, was formulated through the Coefficient Form PDE interface. Appropriate finite element discretization and time-dependent solvers were employed to capture the evolution of the solutions. The obtained numerical simulations successfully reproduced the theoretical predictions, demonstrating the effectiveness and flexibility of COMSOL Multiphysics® for the investigation of nonlinear PDEs with variable exponents.

Theory

Problem 1

We first consider the following initial-boundary problem

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + |u_t|^{p(x)-1} u_t = f_1(u, v), & \text{in } \Omega, t > 0, \\ v_{tt} - \Delta v + m_2^2 v + |v_t|^{r(x)-1} v_t = f_2(u, v), & \text{in } \Omega, t > 0, \\ u(x, t) = v(x, t) = 0, & \text{on } \partial\Omega, t \geq 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & \text{in } \Omega, \\ u_t(x, 0) = u_1(x), \quad v_t(x, 0) = v_1(x), & \text{in } \Omega. \end{cases} \quad (1)$$

where Ω is a bounded and regular domain of R^n ($n=1,2,3$), with a smooth boundary $\partial\Omega$, $f_1(u, v)$ and $f_2(u, v)$ are the source terms defined by

$$\begin{aligned} f_1(u, v) &= a|u + v|^{2(q(\cdot)+1)}(u + v) + b|u|^{q(\cdot)}u|v|^{q(\cdot)+2}, \\ f_2(u, v) &= a|u + v|^{2(q(\cdot)+1)}(u + v) + b|u|^{q(\cdot)+2}|v|^{q(\cdot)}, \end{aligned}$$

where m_1 and m_2 are positive constants, a and b are non-negative constants; and $p(\cdot)$, $r(\cdot)$ and $q(\cdot)$ are given continuous functions on $\bar{\Omega}$ satisfying some conditions.

Problem 2

As the next problem, we consider the following $p(x)$ -biharmonic equation.

Let Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega$. We examine the subsequent initial-boundary value problem

$$\begin{aligned} u_{tt} + \Delta(|\Delta u|^{p(x)-2}\Delta u) + |u_t|^{q(x)-2}u_t &= 0 \\ u(x, t) = \Delta u(x, t) &= 0, \quad \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = \phi(x), \quad u_t(x, 0) &= \psi(x), \quad \text{in } \Omega \end{aligned} \quad (2)$$

Here $\Delta_{p(x)}^2 u = \Delta(|\Delta u|^{p(\cdot)-2}\Delta u)$ and the exponents $p(\cdot)$ and $q(\cdot)$ are given measurable functions on Ω satisfying

$$2 \leq p_1 \leq p(x) \leq p_2 \leq q_1 \leq q(x) \leq q_2 \leq p_1^*$$

with

$$\begin{aligned} p_1 &= \text{ess inf}_{x \in \Omega} p(x) \\ p_2 &= \text{ess sup}_{x \in \Omega} p(x) \\ q_1 &= \text{ess inf}_{x \in \Omega} q(x) \\ q_2 &= \text{ess sup}_{x \in \Omega} q(x) \end{aligned}$$

and

$$p_1^* = \begin{cases} \frac{np_1}{n-p_1}, & \text{if } n > p_1, \\ +\infty, & \text{if } n \leq p_1. \end{cases}$$

Additionally, we assume that $p(x)$ and $q(x)$ satisfy the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq -\frac{A}{\log|x - y|}$$

for all $x, y \in \Omega$, with $|x - y| < \delta$, $A > 0$, $0 < \delta < 1$.

Use of Simulation Apps

For **Problem 1**, the numerical simulations are performed using two time-discretization schemes, namely the Backward Differentiation Formula (BDF) and the Generalized- α (GA) methods. For the spatial discretization, a P1 finite element method is employed.

We reformulated the system of equations (1) into the following wave-equation form in COMSOL Multiphysics®.

$$\begin{aligned} -\Delta W^{n+1} + \mu W^{n+1} &= F \quad \text{in } \Omega, \\ W^{n+1} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

We solve this system using the finite element method (FEM) with the following settings:

1. A time step of $\Delta t = 10^{-4}$ is sufficiently small to capture the finite-time blow-up behavior accurately.

2. The domain Ω is discretized into a triangular mesh comprising 3,160 elements and 12,890 degrees of freedom.

Subsequently, we investigated the evolution of the total energy associated with system (1) throughout the solution. The energy of the system is defined as follows

$$\begin{aligned} E(t) &= \frac{1}{2} (\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2) \\ &\quad + \frac{1}{2} (m_1^2 \|u\|^2 + m_2^2 \|v\|^2) \\ &\quad - \int_{\Omega} F(u, v) dx \end{aligned}$$

Lastly, we established that the solution exhibits finite-time blow-up at T^* , with an upper bound for the blow-up time given by the following inequality

$$T^* \leq \frac{1 - \sigma}{\xi \sigma \Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

For **Problem 2**, we rewrite the nonlinear fourth-order model as a system of second-order equations to make it compatible with the Coefficient Form PDE interface as follows

$$\begin{aligned} u_{tt} + \Delta(|w|^{p(x)-2}w) + |u_t|^{q(x)-2}u_t &= 0 \\ w - \Delta u &= 0 \end{aligned} \quad (3)$$

in $\Omega \times (0, T)$.

We discretize system (3) for u and w using a centered finite difference scheme for u_{tt} and a backward difference scheme for u_t .

Similarly, the energy decay is computed using an integration operator to evaluate the corresponding energy functional over the domain.

Sources

All information regarding the first problem is taken from the paper [1].

Experimental Results

For **Problem 1**, we use the BDF method for time discretization, together with the P1 finite element method for spatial discretization. The numerical simulations confirm that the solutions of the system (1) exhibit finite-time blow-up. Figure 1 presents the initial data u_0 and v_0 , while Figures 2,3,4

display the corresponding numerical solutions at iterations $n=46$, $n=55$, and $n=56$, respectively.

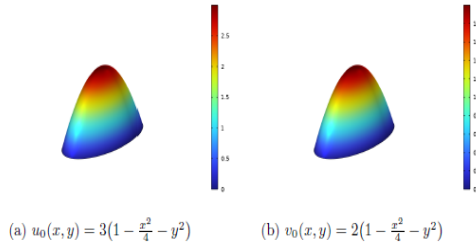


Figure 1: Initial data at $t=0.0$

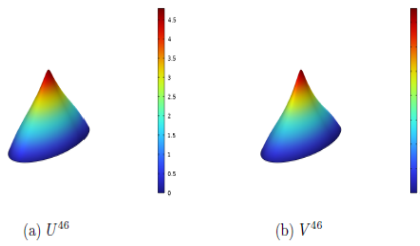


Figure 2: Numerical solutions at $t = 0.0046$

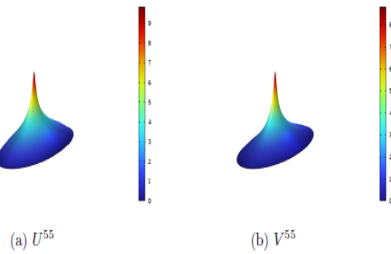


Figure 3: Numerical solutions at $t = 0.0055$

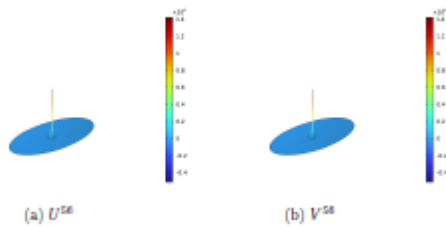


Figure 4: Numerical solutions at $t = 0.0056$, Blow-up time

We repeated the numerical tests using the Generalized- α (GA) method for time discretization. We observed that the solution exhibits blow-up at $t = 0.0062$, showing that the BDF method captures the blow-up behavior faster than the GA method.

We also analyzed the system with negative initial energy and observed that the energy decreases over time, followed by a sudden blow-up of the solution. Table 1 also confirms this behavior with the infinity-norm values reported.

t_n	$\ U\ _\infty$	$\ V\ _\infty$	$E(t_n)$
0.0001	2.00	3.00	-34249.9
0.0032	2.58	3.58	-39458.4
0.0046	3.85	4.85	$-1.5428E+5$
0.0053	6.23	7.23	$-1.7630E+6$
0.0055	18.02	19.03	$-1.134E+7$
0.0056	$1.4E+6$	$1.4E+6$	$-1.386E+47$

Table 1: Numerical values of $\|U\|_\infty$, $\|V\|_\infty$ and $E(t_n)$

In the next analysis, we numerically observed that the solution of Equation (2) tends to zero. The simulations illustrating its evolution are presented in Figures 5–8.

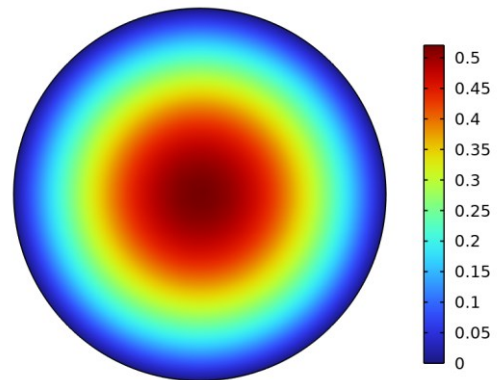


Figure 5: Numerical solutions at $t = 1$

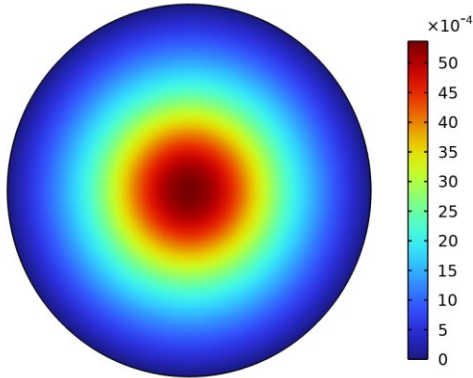


Figure 6: Numerical solutions at $t = 10$

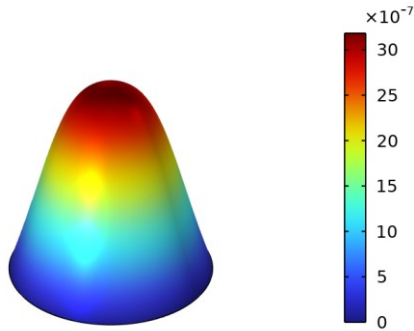


Figure 7: Numerical solutions at $t = 25$

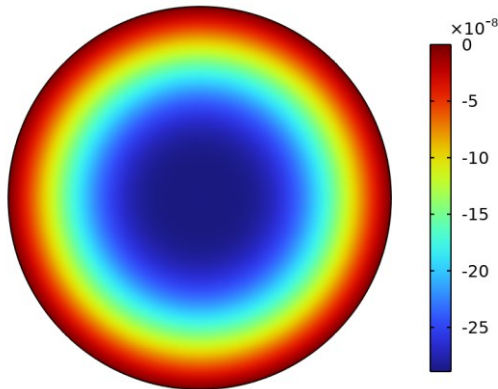


Figure 8: Numerical solutions at $t = 30$

Furthermore, we confirmed the existence of the constant α given in the theoretical part by taking $q(x) = 2$ which guarantees the exponential decay of the energy. In Figure 9, the blue curve shows the energy $E(t)$ for $t \in [0, 30]$, while the green curve represents $y = E(t)e^{0.1t}$.

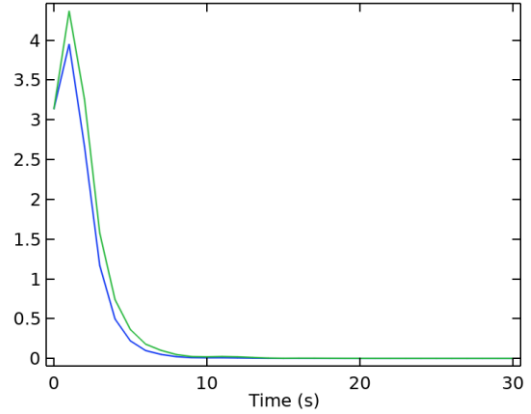


Figure 9: Decay of the energy of the numerical solution

Conclusions

In this study, we numerically investigated two nonlinear partial differential equation models exhibiting distinct dynamical behaviors using COMSOL Multiphysics®. The obtained numerical results demonstrate the effectiveness of COMSOL Multiphysics® in accurately capturing the characteristic dynamics of the considered nonlinear systems..

References

- [1] N. Yılmaz, E. Piskin, B. Okutmustur and S. Mazı Gözen, "A blow-up of solutions for a system of Klein-Gordon equations with variable exponents: Theoretical and Numerical Results," *Mathematica Applicanda*, vol. 51, no. 2, pp. 143-164, 2023.

Acknowledgements

The author gratefully acknowledges the Istanbul Technical University Scientific Research Projects Unit (ITU BAP) for its financial support under Project No. TDK-2023-44744.