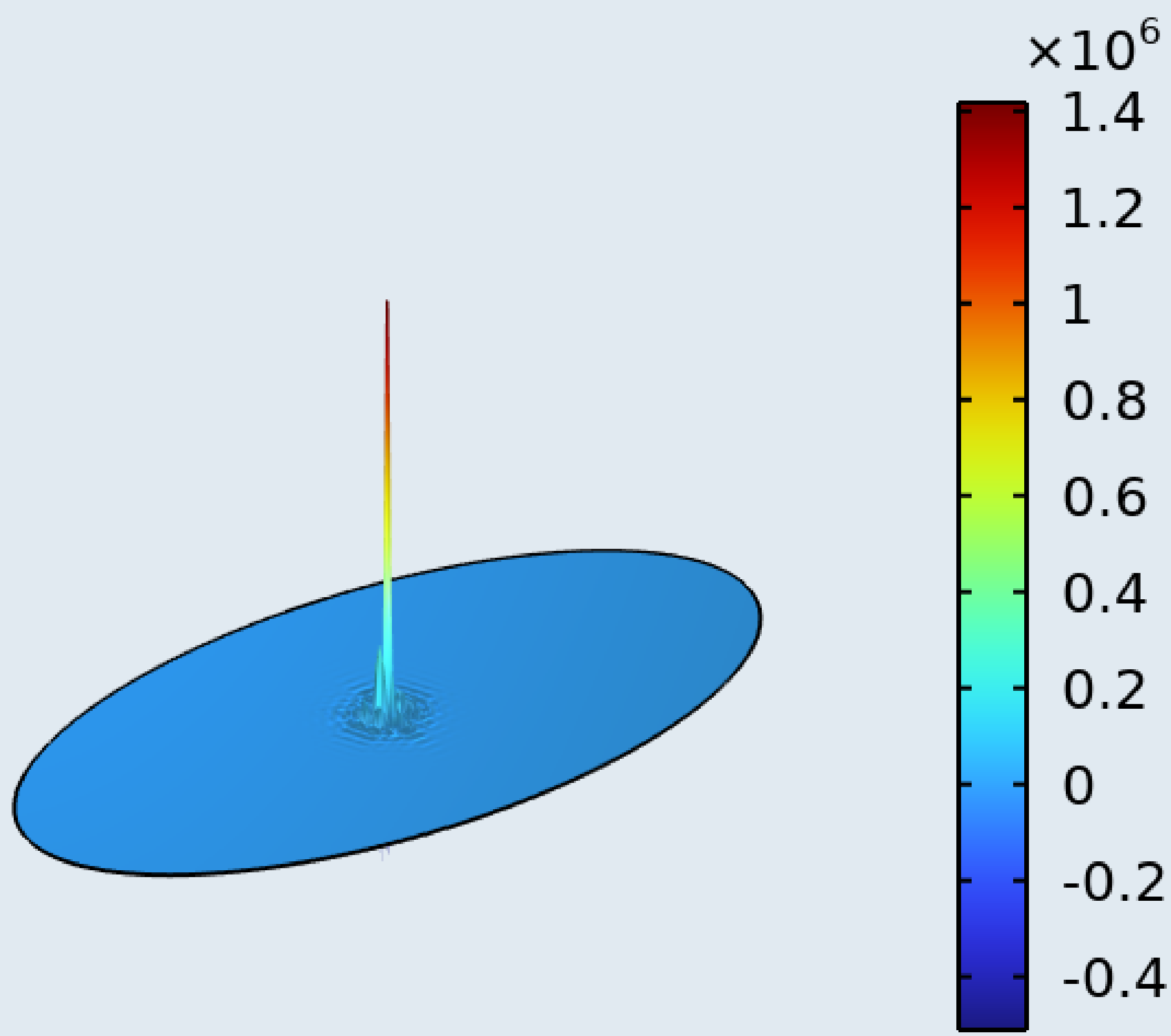




Numerical Investigation of Energy Decay and Blow-up in Variable Exponent PDE Models

Analysis of the dynamical behaviour of two nonlinear PDEs with numerical simulation of blow-up energy and decay results .

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Introduction

Partial differential equations with variable-exponent nonlinearities offer a flexible framework for modeling complex nonlinear phenomena in heterogeneous media, allowing the nonlinear response to vary with spatial position. This spatial variability leads to rich qualitative dynamics and raises important questions concerning the long-time behavior of solutions. In particular, understanding the stability, energy decay, and possible finite-time blow-up of solutions is essential for characterizing the underlying dynamics.

In this study, we first consider a **coupled Klein–Gordon system with a variable exponent**, followed by a **$p(x)$ -biharmonic equation** subject to appropriate initial and boundary conditions. The main objective is to numerically examine the evolution of the solutions and illustrate the corresponding blow-up and energy decay phenomena. The numerical findings are consistent with the theoretical predictions, providing support for the qualitative analysis.

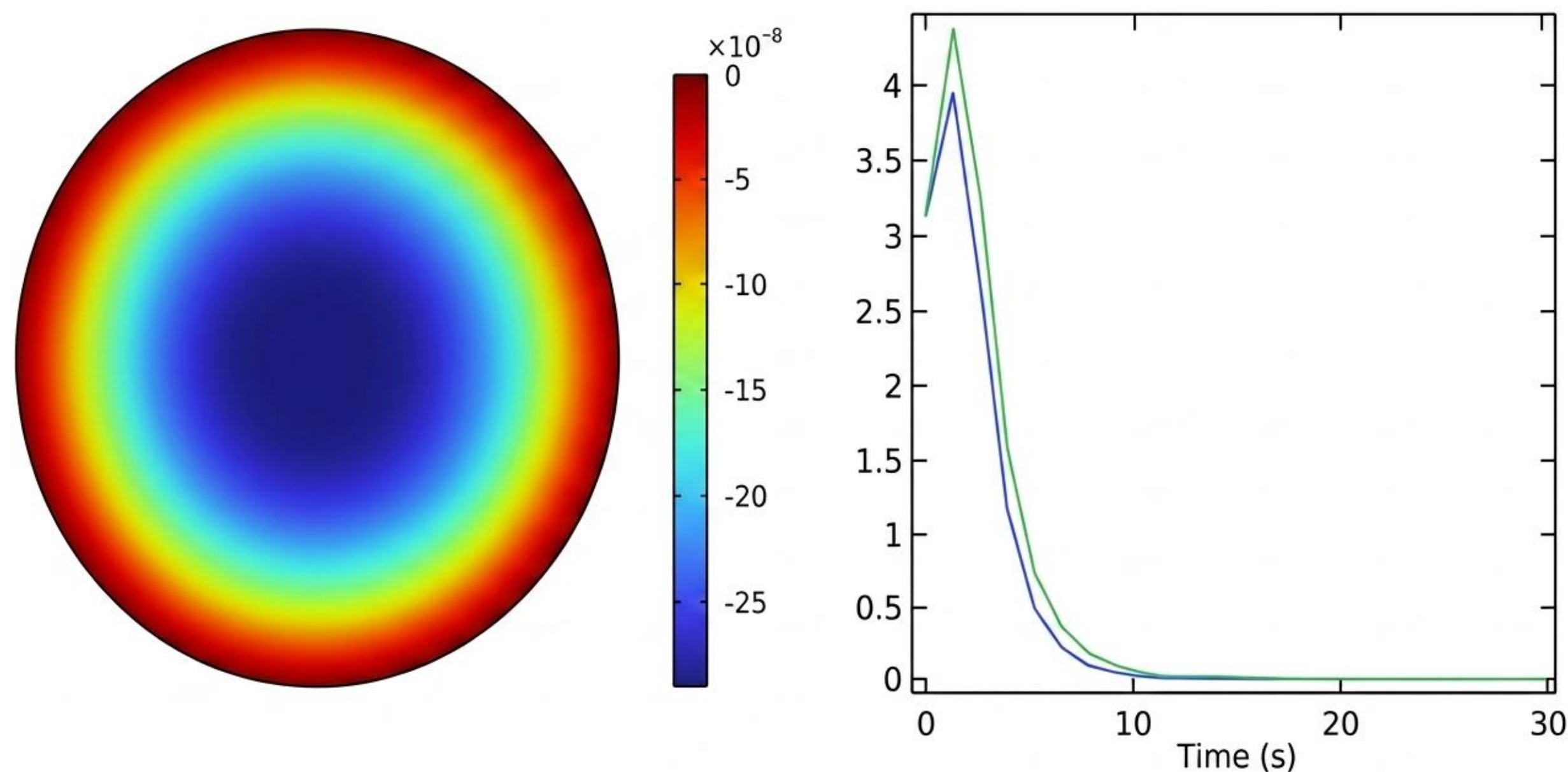


FIGURE 1. Left: Decay of the solution profile. Right: Energy decay $E(t)$ over time .

Methodology

We first consider the following initial–boundary value problem for a coupled system of nonlinear Klein–Gordon equations with variable exponents

$$\begin{aligned} u_{tt} + \Delta(|\Delta u|^{p(x)-2}\Delta u) + |u_t|^{q(x)-2}u_t &= 0, \quad \text{in } \Omega \times (0, T) \\ u(x, t) = \Delta u(x, t) &= 0, \quad \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = \phi(x), \quad u_t(x, 0) &= \psi(x), \quad \text{in } \Omega \end{aligned} \quad (1)$$

The energy of the system is defined as

$$E(t) = \frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx$$

The system is discretized using P_1 Lagrange finite elements in space and backward differentiation formula in time.

Conclusion

The numerical results in Figure 2 show that for the Klein-Gordon system, the **BDF method** captures the finite-time blow-up behavior faster than the **Generalized- α method**. We also observe a decrease in the system's energy and numerically obtain the theoretical blow-up time.

As a second problem, we investigate the decay behavior of the energy associated with the solution of (1). We observe that the energy is bounded from above and tends to zero as time increases, as seen in Figure 1.

For both problems, the consistency between the theoretical and numerical results is demonstrated through simulations performed in **COMSOL Multiphysics®**.

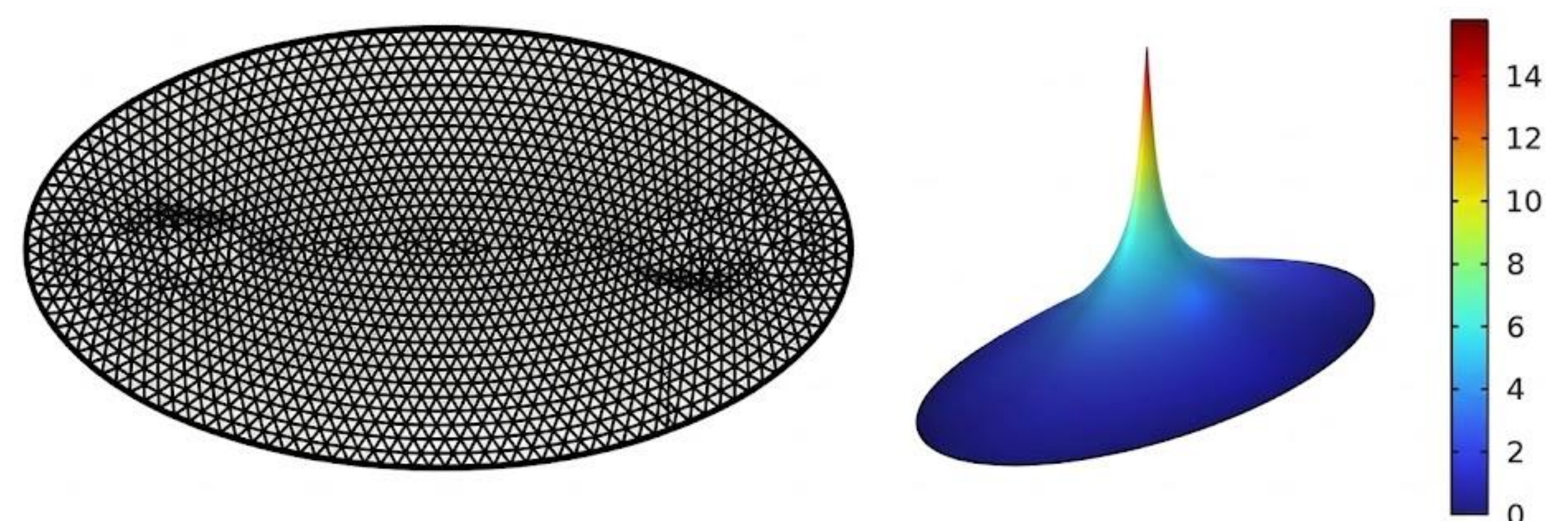


FIGURE 2. Left: Triangular mesh over an elliptical domain. Right: Numerical solution at $t=0.0055$.

REFERENCES

1. Yılmaz, N., Gözen, S. M., Pişkin, E., & Okutmuş, B. (2023). A blow up of solutions for a system of Klein-Gordon equations with variable exponent. Theoretical and Numerical Results.

